Lecture 4

Correction (from last time): ODE

$$v' = g(v), \qquad v(0) = 0$$

$$u(t) = v(t) - tv'(0) \implies u'(t) = v'(t) - v'(0)$$

then we have

$$u' = f(u,t) \implies f(0,0) = 0$$

Theorem 1 (Cauchy-Kovaleskay Theorem in $\mathbb{R}$)**.** *Consider*

$$\partial_n u_i = f_i(x, u, \partial_1 u, ...\partial_{n-1} u) \qquad i = 1, 2..., m$$

with initial data

$$u_i(\check{x}, 0) = \psi_i(\check{x}), \;\; \check{x} \in \mathbb{R}^{n-1}.$$

f_i, ψ_i are analytical at the origin. Then there exists a unique solution u that is analytic at the origin.

$$\exists! f \in G^1_{M,r}(0)$$

proof. WLOG assume $\psi_i = 0$. $\quad [f_i(0) = 0]$.

$$u(x) = \sum_{|\alpha| \geq 0} a_\alpha x^\alpha \implies \partial^\alpha u(0) = \alpha! a_\alpha.$$

$$\partial^\alpha u(0) = 0 \;if\; \alpha_n = 0, \qquad z \in \mathbb{R}^{n+m+(n-1m)} \;where\; z \;argument\; of\; f$$

$$\partial_l \partial_n u_i = \partial_{x_l} f_i(z) + \partial_{u_k} f_i(z) \partial_l u_k + \partial_{\partial_j u_k} f_i(z) \partial_l \partial_j u_k$$

$$\partial^\alpha \partial_n u_i = q_\alpha(\partial_z^\beta f_i(z), \partial^\gamma u_k)$$

where

$$|\beta| \leq |\alpha|, \;\; k = 1...m \qquad |\gamma| \leq |\alpha| + 1 \qquad \gamma_n \leq \alpha_n.$$

$$\begin{aligned} \partial^\alpha \partial_n u_i(0) &= q_\alpha(\partial_z^\beta f_i(0), \partial^\gamma u_k(0)) \\ &= Q_\alpha(\partial_z^\beta f_k(0)) \qquad positive coeff. \end{aligned}$$

Uniqueness comes from a_α derived from derivatives of u from the series. Now for existance: $f_i \ll F_i$
Consider $\partial_n U_i = F_i(x, U, \partial_1 U, ...\partial_{n-1} U), \qquad U_i(\check{x}, 0) = 0$

$$|\partial^\alpha \partial_n u_i(0)| = |Q_\alpha(\partial_z^\beta f_k(0))| \leq Q_\alpha(|\partial_z^\beta f_k(0)|) \leq Q_\alpha(\partial_z^\beta F_k(0)) = \partial^\alpha \partial_n U_i(0).$$

$\implies u_i \ll U_i$ *at* 0. We need to find a "nice" F_i Try:

$$\partial_n U_i = \frac{Mr}{r - \sum x_i - \sum u_k - \sum \partial_j u_k}$$

$u_1 = ... = u_m = U \qquad \partial_l U = \partial_\sigma U$ where $\sum x_i = \sigma, \; \sum u_k = mU, \; \sum \partial_j u_k = (n-1) m \partial_l U.$ we have

$$U' = \frac{Mr}{r - \sigma - mU - (n-1)mU'}$$

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consider a simpler form:

$$\frac{Mr}{r-x-y} = \frac{M}{1-\frac{x+y}{r}} = M(1+(x+y)/r+(x+y)^2/r^2+...) = \sum a_{ij}x^iy^j \qquad a_{ij} \geq Mr^{-i-j}$$

consider a better series

$$\frac{M}{(1-x/r)(1-y/r)} = M(1+x/r+(x/r)^2...)(1+y/r+(y/r)^2...) \qquad a_{ij} = Mr^{-i-j}$$

Going back to our problem

$$U' = \frac{M}{(1-\frac{\sigma+mU}{r})(1-\frac{m(n-1)u'}{r})}$$

$$U' - \frac{m(n-1)}{r}(u')^2 = \frac{M}{1-\frac{\sigma+mU}{r}} \ (constant)$$

by setting LFH= constant we have

$$V - bV^2 = K \implies V = \frac{1-\sqrt{1-4Kb}}{2b}$$

$1-\sqrt{1-4kb} = 1 - bK/2 - ...k^2... = c_1K + c_2K^2 + ...(c_k) \geq 0.$

$$U' = G(U,\sigma)$$

□

Theorem 2 (Identity Theorem)**.** *$f \in C^\omega(\Omega)$. Ω open connected.*
$c \in \Omega$. $\partial^\alpha f(c) = 0 \ \forall\alpha \implies f \equiv 0$ on Ω. In part, if $f = 0$ us a non empty open set of Ω, then $f \equiv 0$.

proof. $\Sigma_\alpha = \{x \in \Omega : \ \partial^\alpha f(x) = 0\}$ relatively closed. (the points map to zero which is a point, implies continuous.. pre image of closed set is closed). $\Sigma = \bigcap_\alpha \Sigma_\alpha$ is rel-closed. But Σ is open, $\Sigma \ni c \implies \Sigma = \Omega$ □

Back to C-K : $u_i \in C^\omega(\Omega)$.

$$v_i = \partial_n u_i \in C^\omega(\Omega)$$

$$w_i(x) = f_i(x, u, \partial_1 u, ...\partial_{n-1}u) \in C^\omega(\Omega)$$

We constructed u_i so that $\partial^\alpha v_i(0) = \partial^\alpha w_i(0) \quad \forall\alpha$.
$\implies v_i \equiv w_i$ *in* Ω. C-K proved.

Extensions

-Higher Order Systems. - $D \subseteq \mathbb{R}^{n-1}$ open with $u_i(\check{x},0) = \psi_i(\check{x}), \check{x} \in D$